1.4. Torsion free connections, metric connections, torsion free and metric connections

1.4.1. Torsion free connections

$\Gamma^a{}_b$ is said to be torsion free if $T^a = 0$ meaning

$$\boxed{de^a + \Gamma^a{}_b \wedge e^b = 0}$$

We can choose any frame we want, use $(e^1, \dots, e^n) = (dx^1, \dots, dx^n)$

$$\underbrace{d(dx^a)}_{=0} + \underbrace{\Gamma^a{}_{bc}\, dx^c \wedge dx^b}_{=\Gamma^a{}_{[bc]}\, dx^c \wedge dx^b} = 0$$

$$\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$$

Conclusion: a connection $\Gamma^a{}_b$ is torsion free if and only if in a holonomic frame it satisfies

$$\boxed{\Gamma^a{}_{bc} = \Gamma^a{}_{cb}} \quad \boxed{e^a = dx^a}$$

$$\boxed{(\nabla_a\nabla_b - \nabla_b\nabla_a) f = 0}$$ any frame

$$\boxed{d(\omega_b e^b) = (\nabla_a \omega_b - \nabla_b \omega_a) e^a \otimes e^b}$$

For a torsion free connection

$$R^a{}_b \wedge e^b = 0$$

$$R^a{}_b = \tfrac{1}{2} R^a{}_{bcd}\, e^c \wedge e^d$$

$$R^a{}_{bcd}\, e^c \wedge e^d \wedge e^b = 0$$

$$\shortparallel$$

$$R^a{}_{[bcd]}\, e^c \wedge e^d \wedge e^b$$

$$\boxed{R^a{}_{[bcd]} = 0 \quad \text{any frame}}$$ 1st Bianchi

$$R^a{}_{bcd} = -R^a{}_{bdc} \Rightarrow \quad R^a{}_{bdc} + R^a{}_{dcb} + R^a{}_{cbd} = 0$$

$$\boxed{\nabla_X Y - \nabla_Y X = [X, Y]}$$

The 2nd Bianchi identity

$$dR^a{}_b + R^a{}_c \wedge R^c{}_b = 0$$

takes the following form

$\nabla_{[e} K^{|b|}{}_{cd]} = 0$

antisymmetrisation of e, c, d (while b excluded)

1.4.2. Metric connections

Given a connection ∇ suppose, there is a symmetric, non-degenerate

$$g = g_{ab}\, e^a \otimes e^b$$

$\exists$ unique g^{ab} s.t. $g^{ab} g_{bc} = \delta^a{}_c$

$g^a{}_b := g^{ac} g_{cb} = \delta^a{}_b$

such that

$$\nabla g = 0 .$$

There exist a frame $(e^1, \dots, e^n)$ s.t.

$$g = g_{ij}\, e^i \otimes e^j , \quad g_{ij} = \text{const}$$

$$0 = \nabla g = (\underbrace{d g_{ab}}_{=0} - \Gamma^c{}_a g_{cb} - \Gamma^c{}_b g_{ac}) e^a \otimes e^b =$$

$$= - \Gamma_{ab} - \Gamma_{ba}$$

$$\boxed{\Gamma_{ab} + \Gamma_{ba} = 0}$$

$\Gamma_{ab} := g_{ac} \Gamma^c{}_b$

in a frame $e^1, \dots, e^n$ s.t. $g_{ij} = \text{const}$

$$R^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b$$

$$R_{ab} = d\Gamma_{ab} + \Gamma_{ac} \wedge \Gamma^c{}_b =$$

$$= -d\Gamma_{ba} - \Gamma_{cb} \wedge \Gamma_a{}^c =$$

$$= -d\Gamma_{ba} - \Gamma_{bc} \wedge \Gamma^c{}_a$$

$$\boxed{R_{ab} = -R_{ba}}$$ in any frame

1.4.3. Torsion free and metric connections.

Consider now a connection ∇ that is both: torsion free and

$$\nabla g = 0$$

where $g = g_{ij} e^i e^j$ is:

$$g_{ij} = g_{ji}, \quad g^{ij} \text{ exists}$$
$$g^{ik} g_{kj} = \delta^i_j$$

Every tensor g that has those properties is said to be a metric tensor.

Proposition. Given a metric tensor, the metric and torsion free connection exists and is unique.

Consider a normalized frame, that is

Consider a [illegible]

such that
$$g = g_{ij}\, e^i \otimes e^j, \quad g_{ij} = \text{const}$$

Denote:
$$de^i + f^i{}_{jk}\, e^j \wedge e^k = 0,$$
$$f^i{}_{jk} = -f^i{}_{kj}$$

Then,
$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} \left(f_{ljk} + f_{jkl} - f_{klj} \right)$$

Consider a holonomic frame $(dx^1, \dots, dx^4)$
$$g = g_{ab}\, dx^a \otimes dx^b,$$
$$g_{ab} = g_{ba}, \text{ functions}$$

Then
$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(g_{db,c} + g_{db,c} - g_{bc,d} \right)$$

The identities satisfied by the Riemann tensor of the metric and torsion free ∇:

$$\mathcal{R}^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b = \frac{1}{2} R^a{}_{bcd}\, e^c \wedge e^d$$
$$= R^a{}_{bcd}\, e^c \otimes e^d$$

$$R_{abcd} := g_{ae} R^e{}_{bcd}$$

$R_{abcd}\, e^a \otimes e^b \otimes e^c \otimes e^4$ is a tensor

The symmetries

the metricity $\Rightarrow$ $$R_{abcd} = -R_{bacd}$$

$$R_{abcd} = -R_{abdc}$$

the torsion freeness $\Rightarrow$ the 1st Bianchi id. $$R_{a[bcd]} = 0$$

Lemma.

$$R_{abcd} = R_{cdab}$$

The differential identities: the 2nd Bianchi

$$D\mathcal{R}^a{}_b = 0 \iff \nabla_{[e} R_{|ab|cd]} = 0$$

antisym

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0$$

The Ricci tensor:

$$R_{ab} := R^c{}_{acb}$$ no metric tensor here

$$R_{ab} = R_{ba}$$ the symmetries used

The Ricci scalar:

$$R := g^{ab} R_{ab}$$ the metric enters here

Consequences of the identities:

$$0 = g^{ea}\left(\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec}\right) =$$

$$= \nabla_a R^a{}_{bcd} + \nabla_c R^a{}_{bda} + \nabla_d R^a{}_{bac}) =$$

$$= \nabla_a R^a{}_{bcd} - \nabla_c R_{bd} + \nabla_d R_{bc}$$

$$\boxed{\nabla_a R^a{}_{bcd} = \nabla_c R_{bd} - \nabla_d R_{bc}}$$

$$\nabla_a R^{ab}{}_{cb} = \nabla_c R^b{}_b - \nabla_b R^b{}_c$$

$$\nabla_a R^a{}_c = \nabla_c R - \nabla_b R^b{}_c$$

$$2\left(\nabla_a R^a{}_c\right) - \nabla_c R = 0$$

$$\boxed{\nabla_a\left(R^a{}_c - \frac{1}{2} R g^a{}_c\right) = 0}$$

$$R_{ab} - \frac{1}{2} R g_{ab} =: G_{ab}$$

the Einstein tensor

The Weyl tensor is the trace free part of R_{abcd}:

$$R_{abcd} = C_{abcd} + \frac{2}{n-2}\left(g_{a[c} R_{d]b} - g_{b[c} R_{d]a}\right) + \\ - \frac{2}{(n-1)(n-2)} R \, g_{a[c} g_{d]b}$$

$$C_{abcd} = -C_{bacd} = C_{cdab}$$

$$C^a{}_{bac} = 0$$

$C^a{}_{bcd}$ is conformally invariant

$$g' = f^2 g, \quad f \text{ - a function}$$

$$C'^a{}_{bcd} = C^a{}_{bcd}$$

$n = 2 \Rightarrow C^a{}_{bcd} = 0$

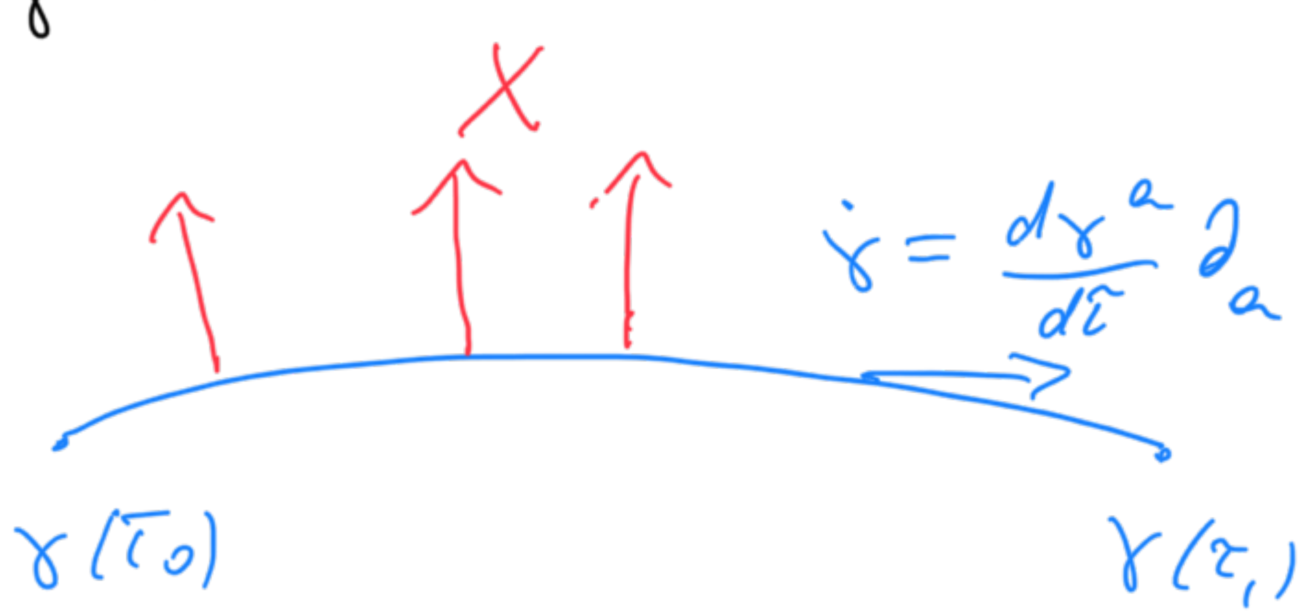
1.4.4. Geodesic curves

Given: a manifold M, a connection ∇

Consider a curve

$$\gamma : [\tau_0, \tau_1] \to M$$

Consider a vector X at each point of γ:



If $\nabla_{\dot\gamma} X = 0$

then X is said to be parallely transported along γ:

$$\nabla_{\dot\gamma} X = \frac{d}{d\tau} X^a + \Gamma^a{}_{bc} X^b \dot\gamma^c = 0$$

γ is said to be geodesic whenever

$$\nabla_{\dot\gamma} \dot\gamma = 0$$

In other words, $\dot\gamma$ is parallely transported.

Suppose ∇ is metric,

$$\nabla_a g_{bc} = 0$$

$$\boxed{\begin{array}{l}\text{Lema (i) } g_{ab}\dot\gamma^a\dot\gamma^b = \text{const along a geodesic curve } \gamma. \\ \text{(ii) } g_{ab}X^a X^b = \text{const along a geodesic curve } \gamma \text{ if } X \text{ is parallely transported along } \gamma.\end{array}}$$

$$\nabla_{\dot\gamma}(g_{ab}\dot\gamma^a\dot\gamma^b) = (\nabla_{\dot\gamma} g_{ab})\dot\gamma^a\dot\gamma^b +$$

$$= g_{ab}\underbrace{(\nabla_{\dot\gamma}\dot\gamma^a)}_{=0}\dot\gamma^b + g_{ab}\dot\gamma^a \underbrace{\nabla_{\dot\gamma}\dot\gamma^b}_{=0}$$

$$\nabla_{\dot\gamma}(g_{ab}\dot\gamma^a X^b) = \nabla_{\dot\gamma} g_{ab}\,\dot\gamma^a X^b + g_{ab}\nabla_{\dot\gamma}\dot\gamma^a X^b +$$

$$+ g_{ab}\dot\gamma^a \nabla_{\dot\gamma} X^b = 0$$

Suppose $\nabla_{\dot\gamma}\dot\gamma = 0$, then either

$$g_{ab}\dot\gamma^a\dot\gamma^b \neq 0 \quad \text{along } \gamma$$
or $$g_{ab}\dot\gamma^a\dot\gamma^b = 0$$

If $g_{ab}\dot\gamma^a\dot\gamma^b \neq 0 \Rightarrow \tau = \tau(\tau')$ s.t.

$$\frac{d\tau}{d\tau'}\frac{d\gamma^a}{d\tau} = \gamma'^a, \quad g_{ab}\gamma'^a\gamma'^b = \pm 1$$

$$\frac{d\tau}{d\tau'} = \frac{1}{\sqrt{|g_{ab}\dot\gamma^a\dot\gamma^b|}} \Rightarrow \tau' = \sqrt{|\dot\gamma_a\dot\gamma^a|}\,\tau$$

proper time/distance

A weaker property:

$$\nabla_{\dot\gamma}\dot\gamma = f\dot\gamma$$

then there exists reparametrization

$$\tau = \tau(\tau')$$

s.t. $\frac{d\tau}{d\tau'}\cdot\dot\gamma =: \gamma'$ satisfies

$$\nabla_{\gamma'}\gamma' = 0$$

Indeed,

$$f(\tau)\cdot\dot\gamma = \nabla_{\dot\gamma}\dot\gamma = \nabla_{\dot\tau'\gamma'}\dot\tau'\gamma' \overset{!}{=} \ddot\tau'\gamma' = \ddot\tau'\cdot\frac{1}{\dot\tau'}\dot\gamma$$

Hence, we are asking about $\tau'(\tau)$:

$$\frac{d^2\tau'}{d\tau^2} = f(\tau)\frac{d\tau'}{d\tau}.$$

On the other hand we know that the result should read

$$d\tau' = \sqrt{|g(\dot\gamma, \dot\gamma)|}\, d\tau.$$

A vector field u is said to be geodesic whenever it satisfies

$$\nabla_u u = 0$$

If

$$g(u,u) \neq 0$$

we can consider

$$u' = \frac{u}{\sqrt{|g(u,u)|}}.$$

It satisfies (let us drop the prime)

$$\nabla_u u = 0, \quad g(u,u) = \text{const}$$

Geodesity is related to extremizing various integrals.

1) The length/time integral

$$\int_{\tau_0}^{\tau_1} d\tau \, |\dot\gamma^a \dot\gamma^b g_{ab}(\gamma(\tau))|^{\frac{1}{2}}$$

is extremized by every $\gamma : \{\tau_0, \tau_1\} \to M$

e.t.

$$\nabla_{\dot\gamma}\dot\gamma = f\dot\gamma\,, \quad f\text{ - any function}$$

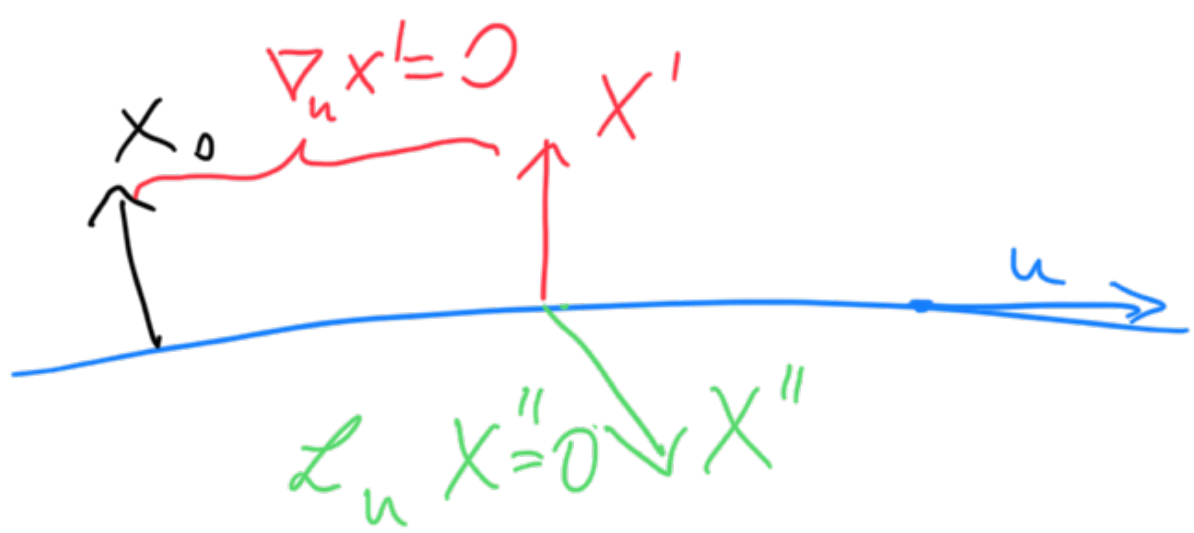
2) The integral

$$\int_{\tau_0}^{\tau_1} \dot\gamma^a \dot\gamma^b g_{ab}(\gamma(\tau))\, d\tau$$

is extremized iff

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

For every $X \in T_pM$ there are two distinct transports along the integral curve of u :

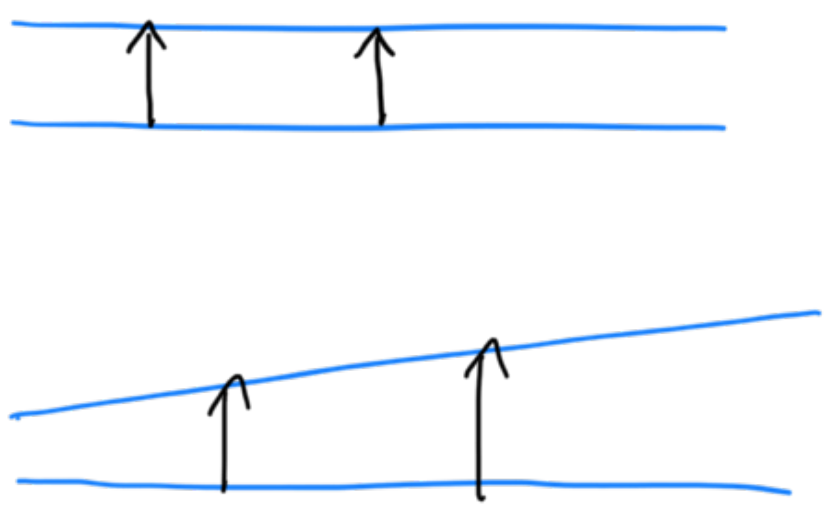


Consider the second case

$$\mathcal{L}_u X = 0$$

Then we may think of X as corresponding to a nearby integral curve

Examples in affine space:



parallel lines

$$\nabla_u X = 0$$

non-parallel lines

$$\nabla_u X = \text{const}$$

$$\nabla_u \nabla_u X = 0$$

Nearby geodesics:

Consider a general case of a family of geodesic curves, integral curves of a v.f. u

$$\nabla_u u = 0, \quad u_a u^a = \text{const}$$

and a nearby geodesic characterised by X s.t.

$$\mathcal{L}_u X = 0$$

The relative velocity is

$$v := \nabla_u X$$

The relative acceleration is

$$a := \nabla_u v = \nabla_u \nabla_u X.$$

Proposition. Suppose $\nabla_u u = 0$, $u_a u^a = \text{const}$ and $\mathcal{L}_u X = 0$. Then

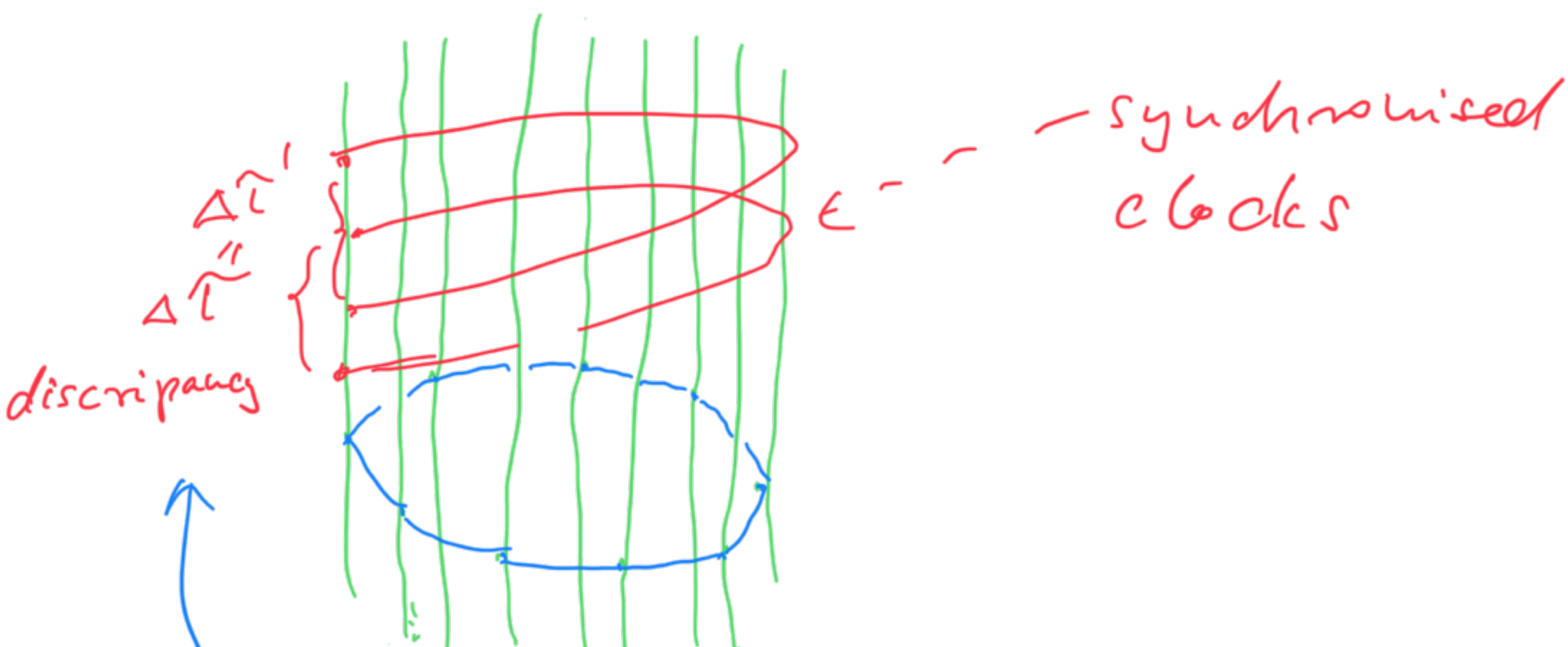
$$u_a X^a = \text{const along the integral curves of } u.$$

$\nabla_u X - \nabla_X u = \mathcal{L}_u X = 0$

Indeed: $\nabla_u (g_{ab} u^a X^b) = g_{ab} u^a \nabla_u X^b = + g_{ab} u^a \nabla_X u^b =$

$$= +\frac{1}{2}\nabla_X (u_a u^a) = 0$$

Spacetime consequence of Proposition. Consider a family of point observers arranged in a circle. Suppose each of them moves along a geodesic curve.



is preserved in time

The discripancy $\Delta\tau$ is given by

$$\int d(u_a e^a) = \Delta\tau$$

Proposition. The relative acceleration is

$$a^a = \nabla_u \nabla_u X = R^a{}_{dcb} u^c u^d X^b$$

Let us see ...

$$u^c \nabla_c u^d \nabla_d X^a = u^c \nabla_c X^d \nabla_d u^a = (u^c \nabla_c X^d) \nabla_d u^a +$$

$$+ u^c X^d \nabla_c \nabla_d u^a = (X^c \nabla_c u^d) \cdot \nabla_d u^a +$$

$$+ u^c X^d \nabla_c \nabla_d u^a =$$

$$= (X^c \nabla_c u^d) \cdot \nabla_d u^a + \underbrace{u^c X^d \nabla_d \nabla_c u^a}_{=} +$$

$$u^c X^d (\nabla \nabla - \nabla \nabla) u^a$$

$+ u \ldots (\nabla_c \nabla_d - \nabla_d \nabla_c) \ldots$

$- (X^d \nabla_d u^c) \nabla_c u^a + X^d \nabla_d u^c \nabla_c u^a$

$= u^c X^d R^a{}_{bcd} u^b$

indeed!

Exercises

1) Suppose the Ricci tensor R_{ab} of a metric tensor g_{ab} satisfies

$$R_{ab} = f g_{ab}.$$

Show, that $f = \text{const}$.

Hint: $\nabla^b \left(R_{ab} - \frac{1}{2} R g_{ab} \right) = 0$

2) Suppose ξ is a Killing vector field of a metric tensor g, that is

$$\mathcal{L}_\xi g = 0.$$

Prove, that for every geodesic curve γ,

$$\xi_a \dot{\gamma}^a = \text{const along } \gamma.$$

Hint: $\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a$

3) Functions of normalised gradient.

Suppose, that f is a function s.t.

$$g^{ab}\nabla_a f \nabla_b f = \text{const}$$

Show that the vector field

$$u^a := g^{ab}\nabla_b f =: \nabla^a f$$

is geodesic, that is

$$\nabla_u u = 0$$

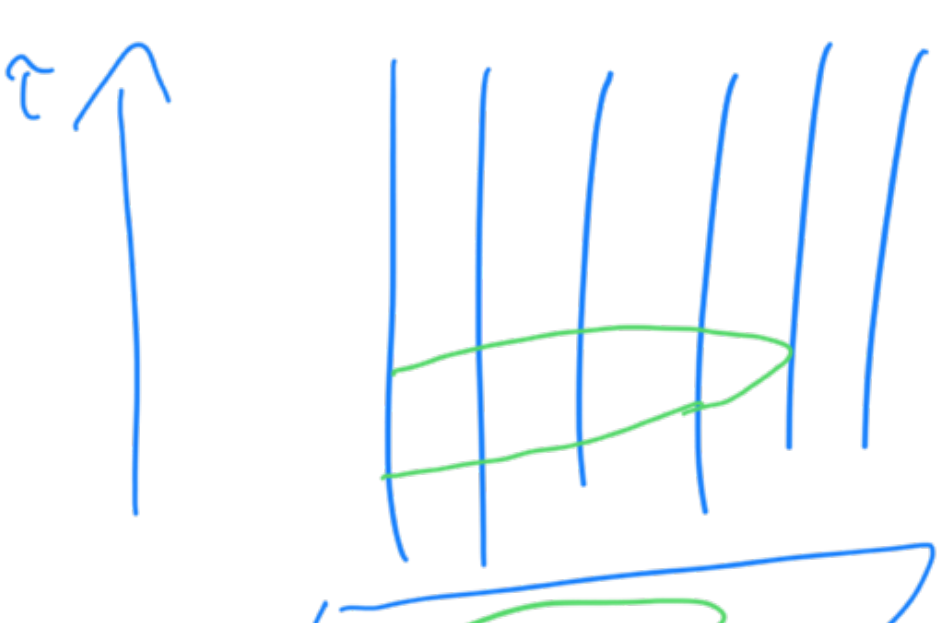
$$\nabla_u u_a = u^b \nabla_b \nabla_a f = g^{bc}\nabla_c f \cdot \nabla_b \nabla_a f =$$

$$= g^{bc}\nabla_c f \nabla_a \nabla_b f = \tfrac{1}{2}\nabla_a (\underbrace{\nabla^b f \cdot \nabla_b f}_{0}) = 0$$

4) Show, that if u is a vector field, then

$$\left.\begin{matrix} \nabla_u u = 0, \\ u^a u^b g_{ab} = \text{const} \end{matrix}\right\} \Rightarrow \mathcal{L}_u u_a = 0$$

5) Synchronization of clocks of geodesic observers.

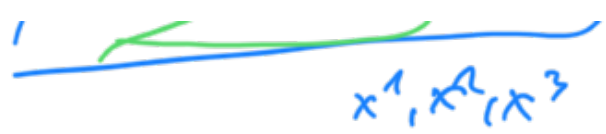


$$\gamma_{x^1,x^2,x^3}(\tau) = \begin{pmatrix} \tau \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\dot{\gamma}_{\vec{x}} = \partial_\tau =: u^a \partial_a$$

$$\dot{\gamma}^a_{\vec{x}} \dot{\gamma}_{\vec{x}\,a} = -1 \Rightarrow u_a dx^a = -d\tau + u_i dx^i$$

$$0 = \mathcal{L}_u (u_a dx^a) = \mathcal{L}_u (-d\tau + u_i dx^i) =$$



$$= \frac{\partial u_i}{\partial \tau} dx^i \Rightarrow u_i = u_i(\vec{x}).$$

$\tau(s), \vec{x}(s)$

synchronization

$$\left(\frac{d\tau}{ds}, \frac{d\vec{x}}{ds}\right) \int_{s_1} u_a dx^a = 0 \Rightarrow -\frac{d\tau}{ds} + u_i \frac{dx^i}{ds} = 0$$

$$\Delta\tau = \int_{s_0} \frac{d\tau}{ds} ds = \int u_i \frac{dx^i}{ds} ds = \oint u_i dx^i =$$

$$= \int d(u_i dx^i)$$

Indeed $\Delta\tau$ is independent of τ.

6) Null surfaces are woven by null geodesics.

Consider a co-dimension 1 surface

$$\mathcal{N} \subset M$$

At every point $p \in \mathcal{N}$ there is a vector

$$n \perp \mathcal{N}$$

Suppose $g_{ab} n^a n^b = 0$. Then $\mathcal{N}$ is called a null surface. Of course n is tangent to $\mathcal{N}$

Show, that

$$\nabla_n n^a = \alpha n^a, \quad \alpha \text{ - a function.}$$

Consider a function f, such that

$$f\Big|_{\mathcal{N}} = \text{const}, \quad df\Big|_{\mathcal{N}} \neq 0 .$$

Notice, that

$$n^a := \nabla^a f$$

is orthogal to $\mathcal{N}$. Indeed

$$X_a n^a = X^a \nabla_a f = X(f) = 0$$

Hence

$$n^a n_a\Big|_{\mathcal{N}} = 0$$

We repeat now the argument that

$$\nabla_n n_a = n^b \nabla_b \nabla_a f = g^{bc} \nabla_c f \cdot \nabla_b \nabla_a f =$$
$$= g^{bc} \nabla_c f \nabla_a \nabla_b f = \frac{1}{2} \nabla_a \left(\nabla^b f \cdot \nabla_b f \right) = \frac{1}{2} \nabla_a (n_b n^b)$$

Let X be tangent to $\mathcal{N}$. Then

$$X^a \nabla_n n_a = \frac{1}{2} X^a \nabla_a (n^b n_b) = 0$$